Integrability

Definition

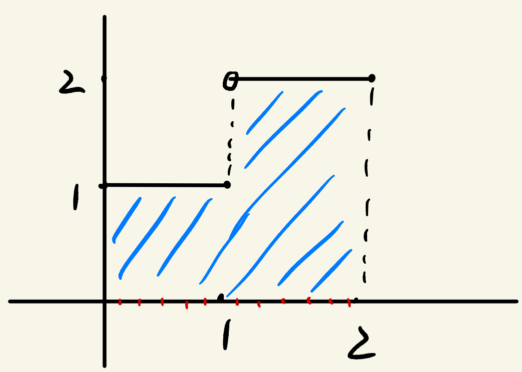
A function $f:[a,b]\to\mathbb{R}$ is called Riemann integrable on $[a,b]$ if there exists a number $L\in\mathbb{R}$ such that for any $\varepsilon>0$, there exists some $\delta>0$ such that if $\dot{P}$ is a tagged partition of $[a,b]$ with $\|\dot{P}\|<\delta$, then $|S(f;\dot{P})-L|<\varepsilon$

Here $\dot{P}:=\{([x_{i-1},x_i],t_i\}_{i=1}^{n}$, $\|\dot{P}\|=\max\limits_{1\le i\le n}\{x_i-x_{i-1}\}$.

$$S(f;\dot{P}):=\sum_{i=1}^{n}f(t_i)(x_i-x_{i-1})$$

Example 1.

$$f(x)=\begin{cases}1, & 0\le x\le 1,\\ 2, & 1<x\le 2.\end{cases}$$



Fix $\varepsilon>0$.

Pick any tagged partition $\dot{P}=\{[x_{i-1},x_i],t_i\}_{i=1}^{n}$ with $\|\dot{P}\|<\delta$.

By Well Ordering Principle, there exists unique $m \in \mathbb{N}$ with $1 \leq m \leq n$ such that $x_{m-1} \leq 1$ and $x_m > 1$.

Let $\dot{P}_1 = \{[x_i, x_{i+1}], t_i\}_{i=1}^{m}$

$\dot{P}_2 = \{[x_i, x_{i+1}], t_i\}_{i=m+1}^{n}$

Clearly, $\|\dot{P}_1\|, \|\dot{P}_2\| < \delta$ and $S(f;\dot{P}) = S(f;\dot{P}_1) + S(f;\dot{P}_2)$.

Note that
$$S(f;\dot{P}_1) = \sum_{i=1}^{m} f(t_i)(x_i - x_{i-1})$$
$$= \sum_{i=1}^{m-1}(x_i - x_{i-1}) + f(t_i)(x_i - x_{i-1})$$

so
$$1 < \sum_{i=1}^{m}(x_i - x_{i-1}) < S(f;\dot{P}_1) < \sum_{i=1}^{m-1}(x_i - x_{i-1}) + 2(x_m - x_{m-1}) < 1 + 2\delta$$

Similarly,
$$S(f;\dot{P}_2) = \sum_{i=m+1}^{n} f(t_i)(x_i - x_{i-1})$$
$$= 2\sum_{i=m+1}^{n}(x_i - x_{i-1})$$

so
$$2(1-\delta) < S(f;\dot{P}_2) < 2$$

Therefore,
$$3 - 2\delta < S(f;\dot{P}) < 3 + 2\delta.$$

Taking $\delta := \frac{\varepsilon}{2}$.

Then for any $\dot{P}$ with $\|\dot{P}\| < \delta$, $|S(f;\dot{P}) - 3| < 2\delta = \varepsilon$.

Hence $\int_0^2 f(x)\,dx = 3$.

Example 2.

$f(x) = x, \quad 0 \le x \le 1.$

Fix $\varepsilon > 0$.

Pick any tagged partition $\dot{P} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$

with $\|\dot{P}\| < \delta$.

Note that $S(f;\dot{P}) = \sum_{i=1}^{n} t_i (x_i - x_{i-1})$ and

$$\frac{x_{i-1}+x_i}{2} - \delta < x_{i-1} \le t_i \le x_i < \frac{x_{i-1}+x_i}{2} + \delta.$$

Then

$$\sum_{i=1}^{n}\left(\frac{x_{i-1}+x_i}{2} - \delta\right)(x_i - x_{i-1}) < S(f;\dot{P}) < \sum_{i=1}^{n}\left(\frac{x_{i-1}+x_i}{2} + \delta\right)(x_i - x_{i-1})$$

$$\sum_{i=1}^{n}\left(\frac{x_{i-1}+x_i}{2} - \delta\right)(x_i - x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) - \delta\sum_{i=1}^{n}(x_i - x_{i-1}) = \frac{1}{2} - \delta$$

$$\sum_{i=1}^{n}\left(\frac{x_{i-1}+x_i}{2} + \delta\right)(x_i - x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2) + \delta\sum_{i=1}^{n}(x_i - x_{i-1}) = \frac{1}{2} + \delta.$$

Taking $\delta = \varepsilon$. For any $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$\left| S(f;\dot{P}) - \frac{1}{2} \right| < \delta = \varepsilon$$

Hence, $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Example 3.

$$f(x) = \begin{cases} \frac{1}{n}, & x = \frac{1}{n}, \\ 0, & \text{otherwise on } [0,1] \end{cases}.$$

Fix $\varepsilon > 0$.

Pick any tagged partition $\dot{P}$ with $\|\dot{P}\| < \delta$.

By AP, there exists some $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{\varepsilon}{2}$$

Let $\dot{P}_1$ be the subset of $\dot{P}$ with tags in $\{1, \dots, \frac{1}{N}\}$ and $\dot{P}_2$ the subset of $\dot{P}$ with tags outside $\{1, \dots, \frac{1}{N}\}$.

Note that
$$\begin{aligned} 0 \le S(f; \dot{P}) &= S(f; \dot{P}_1) + S(f; \dot{P}_2) \\ &\le 2N\delta + \frac{1}{N} \cdot 1 \\ &< 2N\delta + \frac{\varepsilon}{2} \end{aligned}$$

Taking $\delta := \frac{\varepsilon}{4N}$. For any $\dot{P}$ with $\|\dot{P}\| < \delta$,

$$|S(f; \dot{P})| < \varepsilon.$$

Hence $\int_0^1 f(x)\,dx = 0$.

Example 4.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0, 1], \\ 0, & x \in \mathbb{Q}^c \cap [0, 1]. \end{cases}$$

We want there exists $\varepsilon_0 > 0$ such that for any $\delta > 0$, for any $L \in \mathbb{R}$, there exists a tagged partition $\dot{P}$ with $\|\dot{P}\| < \delta$ such that

$|S(f; \dot{P}) - L| > \varepsilon_0$.

Let $\varepsilon_0 = \frac{1}{3}$. For any $L \in \mathbb{R}$, there exists $L' \in (0, 1)$ such that $|L - L'| > \frac{1}{3}$.

For $\delta > 0$, let $P = (x_0, \dots, x_n)$ with $x_m = L'$ and $\|P\| < \delta$.

Pick $t_1, \dots, t_m \in \mathbb{Q}$ and $t_{m+1}, \dots, t_n \in \mathbb{Q}^c$.

Then
$$\begin{aligned} S(f; \dot{P}) &= \sum_{i=1}^{m} f(t_i)(x_i - x_{i-1}) + \sum_{i=m+1}^{n} f(t_i)(x_i - x_{i-1}) \\ &= \sum_{i=1}^{m} (x_i - x_{i-1}) \\ &= x_m - x_0 = L' - 0 = L' \end{aligned}$$

Thus $|S(f; \dot{P}) - L| = |L' - L| > \frac{1}{3} = \varepsilon_0$

$\square$